

Quantum Parallelism in Quantum Information Processing

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We investigate distinguishability (measured by *fidelity*) of the initial and the final state of a qubit, which is an object of the so-called nonideal quantum measurement of the first kind. We show that the fidelity of a nonideal measurement can be greater than the fidelity of the corresponding ideal measurement. This result is somewhat counterintuitive, and can be traced back to the *quantum parallelism* in quantum operations, in analogy with the quantum parallelism manifested in the quantum computing theory. In particular, as the quantum parallelism in quantum computing underlies efficient quantum algorithms, the quantum parallelism in quantum information theory underlies the classically unexpected increase of fidelity.

KEY WORDS: quantum operations; fidelity; nonideal measurement; quantum parallelism.

1. INTRODUCTION: FIDELITY AND QUANTUM MEASUREMENT

One of the main topics in the quantum computer studies (Grover, 1997; Kitaev *et al.*, 1999; Shor, 1997) is the task of storing a single qubit's state (e.g., in the quantum computer's memory) versus the unavoidable influence of the qubit's environment resulting in various kinds of the "quantum operations" on a qubit. Needless to say, the efforts performed in this regard should make implementation of the error-correction strategies and methods easier and hopefully more efficient (DiVincenzo, 1996; Dugić, 2000; Ekert and Macchiavello, 1996; Shor, 1995; Steane, 1996; Viola and Lloyd, 1998; Zanardi and Rasetti, 1997).

The "quantum operations" (Kraus, 1983) generally result in an uncontrollable change of the qubit's state which is characterized by the decrease in *fidelity* (Kraus, 1983; Uhlman, 1976). The latter is a useful measure of distinguishability of the initial and the final qubit's states, yet not representing a metric in the Hilbert state

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space of the qubit. More precisely, fidelity is defined by the following expression (Kraus, 1983; Uhlman, 1976):

$$F(\hat{\rho}, \hat{\sigma}) = F(\hat{\sigma}, \hat{\rho}) = \sqrt{\hat{\rho}^{1/2} \hat{\sigma} \hat{\rho}^{1/2}}. \quad (1)$$

It equals unity if and only if $\hat{\sigma} = \hat{\rho}$ (which is the perfect storage of the qubit's state), while it equals zero for the orthogonal initial and final states. In general, the fidelity satisfies $0 \leq F(\hat{\sigma}, \hat{\rho}) \leq 1$. Above, $\hat{\sigma}$ and $\hat{\rho}$ represent the initial and the final state of the qubit. There is also an alternative characterization of fidelity (Uhlman, 1976) which proves to be equivalent with the definition in Eq. (1).

Here we report on the observation that the fidelity of the so-called *nonideal quantum measurements of the first kind* can result in the *fidelity increase* relative to the fidelity of the corresponding ideal measurements. This result is counterintuitive, for the simple reason that—relative to the ideal measurements (Neumann, 1955)—the nonideal measurements bear *unavoidable uncertainty (ignorance)* in the final state ($\hat{\rho}$) of the qubit (Araki and Yanase, 1960; Wigner, 1952). Needless to say, it is our *classical intuition* which tacitly assumes that lack of information on the system's (qubit's) state should imply decrease of fidelity relative to the situation(s) in which there is no uncertainty. Simultaneously, we classically expect the entropy increase to be manifested with the decrease of fidelity (Lloyd, 1989). We hereby show that the rather unexpected increase of fidelity can be traced back to the *quantum parallelism* in quantum information processing, in full analogy with the quantum parallelism as defined in the quantum computing theory.

In Section 2 we give precise definition of a nonideal measurement, as well as a precise formulation of the task to be performed. In Section 3 we show that a nonideal measurements can lead to the fidelity increase. Section 4 contains discussion of this and related nonclassical phenomena, while the conclusions are given in Section 5.

2. NONIDEAL QUANTUM MEASUREMENTS

A “quantum operation” is defined as the map of the initial system's state $\hat{\sigma}$ (Kraus, 1983):

$$E : \hat{\sigma} \rightarrow \sum_n \hat{A}_n \hat{\sigma} \hat{A}_n^\dagger, \quad (2)$$

where the \hat{A}_n are system operators which satisfy the completeness relation $\sum_n \hat{A}_n \hat{A}_n^\dagger = \hat{I}$; conversely, any map of this form is a quantum operation.

As a special kind of the quantum operations appear the so-called ideal quantum measurements of the first kind (d'Espagnat, 1971; Neumann, 1955), for which

Eq. (2) reads:

$$E : \hat{\sigma} \rightarrow \sum_n \hat{P}_n \hat{\sigma} \hat{P}_n, \tag{3}$$

where the orthogonal projectors \hat{P}_n represent the eigenprojectors of the measured observable, satisfying $\sum_n \hat{P}_n = \hat{I}$.

In the quantum information (and computation) issues, the quantum measurement processes prove to be of substantial importance; e.g., as the (intermediate or the final) steps in quantum computing algorithms, as the procedure of preparation of the qubits' states, or as a formal analogue of the process of decoherence (Dugić, 1996, 1997; Zurek, 1982, 1993), as well as in some quantum information protocols. However, as it was pointed out by Wigner (1952), and later elaborated by Araki and Yanase (1960), the *realistic* quantum measurements usually suffer from *unavoidable errors*, i.e., from unavoidable uncertainty in the final state of the measured object. Let us put this notion in the mathematical terms.

First, without loss of generality, consider the ideal quantum measurement of the observable \hat{S}_z —the z -component of the “spin” (qubit). The ideal measurement of \hat{S}_z is presented by (Neumann, 1955):

$$\hat{U}|\uparrow\rangle|\chi\rangle = |\uparrow\rangle|+\rangle, \tag{4}$$

where \hat{U} represents the unitary (Schrödinger) evolution in time of the combined system “object (qubit) + apparatus (Q + A),” the initial state $|\uparrow\rangle$ is the eigenstate of \hat{S}_z for the value $+\hbar/2$, for arbitrary initial state of the apparatus $|\chi\rangle$ and we omit unnecessary symbol of the tensor product. Similarly, for the initial state of the object $|\downarrow\rangle$ which is the eigenstate of \hat{S}_z for the value $-\hbar/2$, the ideal measurement is defined as:

$$\hat{U}|\downarrow\rangle|\chi\rangle = |\downarrow\rangle|-\rangle, \tag{5}$$

while $\langle +|-\rangle = 0$.

However, as it was emphasized by Wigner (1952), these expressions refer directly *only* to the quantum measurements of the *constants of motion*. Following Wigner (1952), Araki and Yanase (1960) showed that the quantum measurements of the observables which are not the constants of motion are possible, but only with the limited accuracy. Actually, for the *nonideal quantum measurement* of \hat{S}_z one obtains (we introduce the normalization factors in the original expressions (Araki and Yanase, 1960)):

$$\hat{U}|\uparrow\rangle|\chi\rangle = (1 - \epsilon_\uparrow^2)^{1/2}|\uparrow\rangle|+\rangle + \epsilon_\uparrow|\downarrow\rangle|-\rangle, \tag{6}$$

$$\hat{U}|\downarrow\rangle|\chi\rangle = (1 - \epsilon_\downarrow^2)^{1/2}|\downarrow\rangle|-\rangle + \epsilon_\downarrow|\downarrow\rangle|+\rangle, \tag{7}$$

for the initial states considered above, respectively. Subsequently Yanase (1961)

was able to show that:

$$\epsilon_{\uparrow}^2 + \epsilon_{\downarrow}^2 = \epsilon^2 \geq (8\|\hat{M}_x\|^2)^{-1} \quad (8)$$

where \hat{M}_x represents an additive constant of motion of the apparatus; note that, as the apparatus becomes more macroscopic, the lower bound of ϵ becomes smaller (Araki and Yanase, 1960).

Now, relative to the ideal measurements presented by Eqs. (4) and (5), the nonideal measurements introduce *unavoidable error* ϵ in knowing the value of the measured quantity. Actually, as it directly follows from, e.g., Eq. (6), the object's final state reads:

$$\hat{\rho}' = \text{tr}_A[\hat{U}|\uparrow\rangle\langle\uparrow| \otimes |\chi\rangle\langle\chi| \hat{U}^\dagger] = (1 - \epsilon^2/2)|\uparrow\rangle\langle\uparrow| + \epsilon^2/2|\downarrow\rangle\langle\downarrow|, \quad (9)$$

where we have used the equality $\epsilon_{\uparrow} = -\epsilon_{\downarrow} = \epsilon/\sqrt{2}$, which follows from the expression Eq. (8) and from the normalization condition $\langle\uparrow|\langle\chi|\hat{U}\hat{U}^\dagger|\downarrow\rangle\langle\chi| = 0$. With “ tr_A ” we denote the “tracing out” of the apparatus degrees of freedom. Physically, this error is substantial: albeit the object is in an eigenstate of the measured observable (the eigenvalue is $\hbar/2$), the measurement leads to the opposite (wrong) result, giving the value $-\hbar/2$ with nonzero probability $\epsilon^2/2$. From the *information-theoretic point of view*, this error introduces *unavoidable uncertainty (ignorance)* about the object's final state (i.e., instead of the “pure” state $|\uparrow\rangle\langle\uparrow|$, the final state is the mixed state given by Eq. (9)), which is *classically expected to give rise to the fidelity decrease*, relative to the fidelity of the ideal measurement for which $\epsilon = 0$.

Surprisingly enough, we will show that this is not necessarily the case. Actually, we will show that the fidelity of the nonideal quantum measurement can *increase*, giving therefore a counterintuitive result: having less control on the final state of the system, one obtains better fidelity of the operation considered. In the context of the task of storing qubit's state(s), this result is of rather obvious interest.

3. NONIDEAL MEASUREMENTS CAN INCREASE THE FIDELITY

Let us first consider the cases studied in Section 2. And let us introduce the indices “*id*” and “*nonid*” for the ideal and nonideal measurements, respectively.

From the expressions of Eqs. (4) and (5) for the ideal measurement of \hat{S}_z it obviously follows that the final and the initial state are equal, $\hat{\rho} = \hat{\sigma}$ which gives rise—for both expressions (4) and (5)—to the maximum fidelity $F(\hat{\sigma}, \hat{\sigma}) = 1$. However, for the nonideal measurements presented by Eqs. (6) and (7), the initial and the final states are not equal anymore. Actually, e.g., from Eq. (6), it follows that the final state of the qubit is given by Eq. (9), thus giving rise to the fidelity of the measurement:

$$F_{\text{nonid}} \equiv F(\hat{\rho}_{\text{nonid}}, \hat{\sigma}) = \langle\uparrow|\hat{\rho}_{\text{nonid}}|\uparrow\rangle^{1/2} = (1 - \epsilon^2)^{1/2}, \quad (10)$$

where we have used the symmetry property of fidelity (cf. Eq. (1)) and the fact that the initial state $\hat{\sigma}$ is the “pure” quantum state, $\hat{\sigma} = |\uparrow\rangle\langle\uparrow|$. Now, for the difference of the two fidelities we obtain:

$$\Delta F \equiv F_{id} - F_{nonid} \cong \epsilon^2/2 > 0, \quad (11)$$

as one would *classically expect*.

Let us now consider arbitrary initial “pure” state of the qubit:

$$|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \quad |\alpha|^2 + |\beta|^2 = 1, \quad (12)$$

and let us calculate the fidelities of the ideal and nonideal measurements of \hat{S}_z .

For the ideal measurement, the expressions Eqs. (4), (5) give for the final state of the combined system $Q + A$:

$$\hat{U}|\Psi\rangle|\chi\rangle = \alpha|\uparrow\rangle|+\rangle + \beta|\downarrow\rangle|-\rangle, \quad (13)$$

which gives for the final state of the qubit:

$$\hat{\rho}'_{id} = |\alpha|^2|\uparrow\rangle\langle\uparrow| + |\beta|^2|\downarrow\rangle\langle\downarrow|. \quad (14)$$

Now, the fidelity of the ideal measurement reads:

$$F'_{id} = \langle\Psi|\hat{\rho}'_{id}|\Psi\rangle^{1/2} = [1 - 2|\alpha|^2 + 2(|\alpha|^2)^2]^{1/2}. \quad (15)$$

On the other side, for the nonideal measurement, after some simple algebra, one obtains for the final state of the combined system:

$$\hat{U}|\Psi\rangle|\chi\rangle = [\alpha(1 - \epsilon_{\uparrow}^2)^{1/2} + \beta\epsilon_{\downarrow}]|\uparrow\rangle|+\rangle + [\beta(1 - \epsilon_{\downarrow}^2)^{1/2} + \alpha\epsilon_{\uparrow}]|\downarrow\rangle|-\rangle, \quad (16)$$

which after the tracing out gives for the final state of the qubit:

$$\hat{\rho}'_{nonid} = |\alpha(1 - \epsilon_{\uparrow}^2)^{1/2} + \beta\epsilon_{\downarrow}|^2|\uparrow\rangle\langle\uparrow| + |\beta(1 - \epsilon_{\downarrow}^2)^{1/2} + \alpha\epsilon_{\uparrow}|^2|\downarrow\rangle\langle\downarrow|. \quad (17)$$

The corresponding fidelity reads:

$$F'_{nonid} = \langle\Psi|\hat{\rho}'_{nonid}|\Psi\rangle^{1/2} = \{|\alpha|^2 \cdot |\alpha(1 - \epsilon^2/2)|^{1/2} - 2^{-1/2}\beta\epsilon\}^2 + |\beta|^2 \cdot |\beta(1 - \epsilon^2/2)|^{1/2} + 2^{-1/2}\alpha\epsilon\}^2. \quad (18)$$

To simplify the expression Eq. (18) we treat complex numbers α and β as the vectors in plane, so defining the angle θ as $\cos\theta = \bar{\alpha} \cdot \beta / |\alpha| \cdot |\beta|$. Then Eq. (18) reads:

$$F'_{nonid} = \{1 - \epsilon^2/2 + 2(|\alpha|^2)^2 - 2(|\alpha|^2)^2\epsilon^2 + 2|\alpha|^2\epsilon^2 - 2|\alpha|^2 \\ - \epsilon|\alpha|^3[8(1 - |\alpha|^2)(1 - \epsilon^2/2)]^{1/2} \cos\theta \\ + \epsilon|\alpha|[2(1 - |\alpha|^2)(1 - \epsilon^2/2)]^{1/2} \cos\theta\}^{1/2}. \quad (19)$$

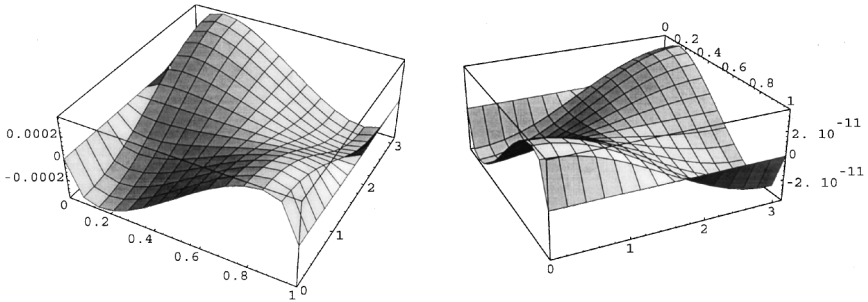


Fig. 1. The plot of $F_{id}^2 - F_{nonid}^2$ against $(|\alpha|^2, \theta)$, for the respective intervals $[0, 1]$ and $[0, \pi]$, for the two values of ϵ (note the different orientations of the figures): (a) $\epsilon = 10^{-3}$, and (b) $\epsilon = 10^{-10}$. The values of the variables are in the horizontal plane. The shape of the plot is independent on the values of ϵ : In the both cases appears that the given difference (and also the fidelity difference $F'_{id} - F'_{nonid}$ is *negative* for the following combinations of the intervals: $\{|\alpha|^2 \in (0, 0.5) \text{ and } \theta \in (0, \pi/2)\}$, as well as for $\{|\alpha|^2 \in (0.5, 1) \text{ and } \theta \in (\pi/2, \pi)\}$. The maximum (minimum) value of the difference is of the order of 10ϵ (0.1ϵ).

From Eqs. (15) and (19) we obtain:

$$\begin{aligned}
 F_{id}^{\prime 2} - F_{nonid}^{\prime 2} &= \epsilon^2/2 + 2(|\alpha|^2)^2\epsilon^2 - 2|\alpha|^2\epsilon^2 \\
 &\quad + \epsilon|\alpha|^3[8(1 - |\alpha|^2)(1 - \epsilon^2/2)]^{1/2} \cos \theta \\
 &\quad - \epsilon|\alpha|[2(1 - |\alpha|^2)(1 - \epsilon^2/2)]^{1/2} \cos \theta.
 \end{aligned}
 \tag{20}$$

Keeping in mind positivity of fidelity, the fidelity difference, $F'_{id} - F'_{nonid}$, is of the same sign as the difference given by Eq. (20).

In Fig. 1 we give the plot of $F_{id}^2 - F_{nonid}^2$ against $(|\alpha|^2, \theta)$, for the respecting intervals $[0, 1]$ and $[0, \pi]$, for the two values of ϵ . As it is obvious from Fig. 1, the shape of the plot is independent on the values of ϵ , while the maximum (minimum) value(s) of the difference is of the order of 0.1ϵ (10ϵ).

From Fig. 1 we conclude that our classical intuition, which is confirmed by Eq. (11), is also confirmed by Fig. 1, *but only for special choice* of $|\alpha|^2$ and θ . Actually, as it is obvious from Fig. 1, one has, for instance:

$$F'_{nonid} > F'_{id}, \quad |\alpha|^2 \in (0, 0.5) \quad \text{and} \quad \theta \in (0, \pi/2),
 \tag{21}$$

which challenges our classical intuition.

4. INTERPRETATION OF THE FIDELITY INCREASE

The quantum operations (nonideal measurements) presented by Eqs. (6) and (7), are “coherently superimposed” in the operation presented by Eq. (16). That is, due to the linearity of the Schrödinger law, the two mutually independent operations in Eqs. (6) and (7) are coherently superimposed in Eq. (16) This gives

rise to the fidelity increase as presented by Eq. (21). This observation calls for *analogy* with the *quantum parallelism* as defined in the quantum computation theory (Grover, 1997; Kitaev *et al.*, 1999; Shor, 1997). But this interpretation leads to the following question: whether the fidelity increase can or cannot ever be observed for the arbitrary initial “mixed state” (Neumann, 1955; d’Espagnat, 1971) (“incoherent mixture” of the initial states $|\uparrow\rangle$ and $|\downarrow\rangle$) of the qubit?

To answer this question we consider arbitrary initial “mixed state”:

$$\hat{\sigma} = W_1|\uparrow\rangle\langle\uparrow| + W_2|\downarrow\rangle\langle\downarrow|, \quad W_1 + W_2 = 1, \tag{22}$$

and calculate the fidelities of both ideal and nonideal measurement of \hat{S}_z ; note that for $W_1 = 1$ ($W_2 = 1$) one obtains the case(s) studied above, i.e., the expression(s) Eq. (4) (Eq. (5)).

Again, as it can be easily shown, the ideal measurement of \hat{S}_z does not change the initial state of the qubit, thus giving rise to the maximum fidelity. However, for the nonideal measurement, after some algebra one obtains for the final state of the qubit:

$$\begin{aligned} \hat{\rho}'' = \text{tr}_A[\hat{U}\hat{\sigma}|\chi\rangle\langle\chi|\hat{U}^\dagger] &= [W_1(1 - \epsilon^2/2) + W_2\epsilon^2/2]|\uparrow\rangle\langle\uparrow| \\ &+ [W_2(1 - \epsilon^2/2) + W_1\epsilon^2/2]|\downarrow\rangle\langle\downarrow|. \end{aligned} \tag{23}$$

Therefore, fidelity in this case reads:

$$\begin{aligned} F''_{\text{nonid}} &= \text{tr}\{\hat{\rho}'_{\text{nonid}}{}^{1/2}\hat{\sigma}\hat{\rho}'_{\text{nonid}}{}^{1/2}\}^{1/2} = \text{tr}\{\hat{\sigma}\hat{\rho}'_{\text{nonid}}\}^{1/2} \\ &= W_1\{1 - \epsilon^2/2 + (1 - W_1)\epsilon^2/2W_1\}^{1/2} + (1 - W_1) \\ &\quad \times \{1 - \epsilon^2/2 + W_1\epsilon^2/2(1 - W_1)\}^{1/2} \leq 1, \end{aligned} \tag{24}$$

where we used $[\hat{\sigma}, \hat{\rho}'_{\text{nonid}}] = 0$, and we perceive the last inequality as obvious, while equality refers only to $W_1 = W_2 = 1/2$. It is worth emphasizing that for $W_1 = 1$ ($W_2 = 1$), the expression in Eq. (24) reduces to Eq. (10).

Therefore, for the “mixed” state $\hat{\sigma}$, one *never obtains the fidelity increase*, which justifies that the fidelity increase is *ultimately* due to the coherent superpositions of the qubit’s states, Eq. (12) (and also due to linearity of the Schrödinger law—cf. Eq. (16)). In other words, the classically unexpected fidelity increase is caused by the quantum coherence, i.e., to the parallel quantum operations in the coherent mixtures of the qubit’s states.

5. DISCUSSION

The main result of this paper is the fidelity increase due to nonideal quantum measurements, as pointed out by Eq. (21). The interpretation given in the previous section allows for analogy with the quantum parallelism as distinguished in the quantum computing theory. To this end one may note that, while the quantum

parallelism is a general feature of quantum computation (Grover, 1997; Kitaev *et al.*, 1999; Shor, 1997), the fidelity increase we point out refers to the limited case of the pure state, cf. Eq. (21). Then the following reasoning may seem plausible: since the fidelity increase does not refer to an arbitrary “pure” state $|\Psi\rangle$, the interpretation given in Section 4 should be considered incomplete, if not incorrect. But, we believe this conclusion to be wrong.

Instead, we maintain that our interpretation is of the general validity (i.e., applicable to any “pure” state of the qubit), *while not requiring* the general increase of fidelity for nonideal measurements. That is, the operations considered do not require the fidelity increase per se. Rather, the fidelity increase should be considered *to point to* the quantum parallelism, being its (classically unexpected) consequence. Again, we can make analogy with the quantum parallelism in quantum computing: quantum parallelism in quantum computing does not a priori guarantee that any quantum-computation algorithm will be substantially more efficient than its classical counterpart per se. However, it is well-known that the major motivation for research in quantum computing is exactly the fact that specific implementations of a concrete quantum-computation algorithm can be shown to be more efficient than any possible classical analogue. This, we believe, justifies the *full analogy* between the quantum parallelism in quantum computing, and in the quantum-measurement-like quantum operations (e.g., decoherence) on a qubit.

The following question may at first sight seem reasonable: does the fidelity increase can be used for achieving fidelity arbitrarily close to unity, by choosing a “sufficiently nonideal” measurement, i.e., by choosing sufficiently big ϵ ? But this question hides a misinterpretation of our result. In the presumed case one would have, at least as to the lower bound of ϵ (cf. Eq. (8)) to be a reasonable fraction of unity. However, as the r.h.s. of Eq. (8) implies, the norm of the (bounded) observable \hat{M}_x would have to be of the order of unity, which is physically unacceptable. Actually, such observable would refer to a *microscopic* system (here: apparatus), in contradistinction to the requirement that the apparatus should be sufficiently macroscopic (Araki and Yanase, 1960, 1961; d’Espagnat, 1971; Neumann, 1955; Zurek, 1982). In other words, the use of the nonideal measurements in approaching the equality $F(\hat{\sigma}, \hat{\rho}) \cong 1$ would contradict applicability of the formulae used in the above calculations (which refer only to the “macroscopic apparatus”).

Finally, our considerations bear full generality due to: (i) all the results concerning the measurements of \hat{S}_z can be straightforwardly applied to the measurements of arbitrary observable of a qubit, (ii) the quantum parallelism can be observed only in comparison of the results (i.e., of the fidelities) for the “coherent” (12) and for the “incoherent” (22) mixtures of the *same states*, which is the reason we do not calculate the fidelity for arbitrary mixed state of the qubit, and (iii) the considerations can be straightforwardly extended to account for the N -qubit systems. The latter circumstance can undoubtedly be of some practical interest in quantum information research.

6. CONCLUSION

It is expectable that the lack of information (uncertainty) about a system's state should be characterized by the fidelity decrease, relative to the situations in which there is no uncertainty. In the context of the quantum measurement process, which is a special kind of the so-called "quantum operations," this classical expectation indicates that fidelity of a nonideal quantum measurement should exhibit decrease, relative to the corresponding ideal measurement. However, and *contrary to this classical intuition*, we show that the nonideal measurements can lead to the *fidelity increase*. That is, the fidelity of the nonideal measurements can be greater than the fidelity of the corresponding ideal measurements. This *counterintuitive* result can be traced back to the *quantum parallelism* in the quantum information processing, in full analogy with the quantum parallelism as conventionally discussed in the quantum computing theory. One may note that, as the quantum parallelism underlies the efficient quantum computing algorithms, the quantum parallelism underlies the classically unexpected increase of fidelity of the nonideal quantum measurements.

REFERENCES

- Aharonov, D. (1998). in *Annual Reviews of Computational Physics, Vol. VI*. D. Stauffer, ed., World Scientific, Singapore.
- Araki, H. and Yanase, M. M. (1960). *Physical Review* **120**, 622.
- d'Espagnat, B. (1971). *Conceptual Foundations of Quantum Mechanics*, Benjamin, Reading, MA.
- DiVincenzo, D. P. and Shor, P. W. (1996). *Physical Review Letters* **77**, 3260.
- Dugić, M. (1996). *Physica Scripta* **53**, 9.
- Dugić, M. (1997). *Physica Scripta* **56**, 560.
- Dugić, M. (2000). *Quantum Computers and Computing* **1**, 102.
- Ekert, A. and Macchiavello, C. (1996). *Physical Review Letters* **77**, 2585.
- Grover, L. K. (1997). *Physical Review Letters* **79**, 325.
- Kitaev, A. Yu., Shen', A. Kh., and Vyal'yi, M. N. (1999). *Classical and Quantum Computation*, MCCME, CheRo, Moscow. (in Russian).
- Kraus, K. (1983). *States, Effects, and Operations*, Springer-Verlag, Berlin.
- Lloyd, S. (1989). *Physical Review A* **39**, 5378.
- Shor, P. W. (1995). *Physical Review A* **52**, 2943.
- Shor, P. W. (1997). *SIAM Journal on Computing* **26**, 1484.
- Steane, A. M. (1996). *Physical Review Letters* **77**, 793.
- Uhlman, A. (1976). *Rep. Prog. Math. Physics* **9**, 273.
- Viola, L. and Lloyd, S. (1998). *Physical Review A* **58**, 2733.
- von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton.
- Wigner, E. P. (1952). *Zeitschrift für Physik* **133**, 101.
- Yanase, M. M. (1961). *Physical Review* **123**, 666.
- Zanardi, P. and Rasetti, M. (1997). *Physical Review Letters* **77**, 3305.
- Zurek, W. H. (1982). *Physical Review D* **26**, 1862.
- Zurek, W. H. (1993). *Progress of Theoretical Physics* **89**, 281.